SOCIAL WELFARE FUNCTION AND SOCIAL INDIFFERENCE CURVES

BY

KEN-ICHI INADA

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INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
Applied Mathematics and Statistics Laboratories
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Ken-ichi Inada
Stanford and Tokyo Metropolitan Universities

1.. The purpose of this paper is to clarify the economic implications of Arrow's impossibility theorem of the welfare function [1], and to prove another impossibility theorem. As Arrow's theorem is stated in a very general form, it is applied not only to the economic choice but also to the political choice. On the other hand, such generality may sometimes conceal the direct applicability of the theorem to the problems in some special fields. At first glance, Arrow's theorem seems to have no relevance to the choice under a certain market situation. His theorem concerns the *intrinsic* inconsistency of social choice. But as will be seen below, it has a close relationship to the economic choice under a certain market situation. The social welfare function is a rule through which the social choice function is constructed from individuals' choice functions. In Arrow's theorem, the domain of definition of the social choice function consists of *all subsets of alternatives*. In such a case, there is no social welfare function which satisfies Arrow's conditions 1-5. But, if the domain of definition of the social choice function contains *only some subsets* of alternatives, there may exist some social welfare functions which have the

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desired properties. Arrow's theorem is stated in such a general form that there may be no meaningful way to restrict the domain of definition of the choice function. To do so, we should have to rewrite Arrow's theorem in a more specified form using the terms of traditional economics. We, therefore, assume that a social state can be expressed by a point in the nonnegative orthant of an n-dimensional space. Generally, a choice function is defined on a class of some subsets in this orthant. In Arrow's case, the choice functions are defined on a class of all subsets in this orthant (Fig. 1), and the inconsistency of a non-trivial welfare function is shown for such choice functions. But, if the choice functions are defined on a class of some limited type of subsets, then the inconsistency may be removed. This is our problem. From an economic view point, the most important and meaningful case may be the case where the definition domain of the choice functions is restricted to the class of the budget constraint sets (Fig. 2).

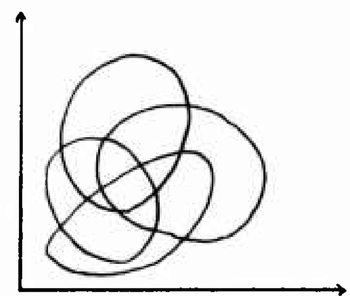

Fig. 1

The difference of the definition domain of the choice functions may entail a different result.[1/] It will be shown that the inconsistency of a non-trivial welfare function remains even if the domain of definition of the choice function is restricted to the class of the budget constraint sets under the market situation of complete price differential. Such a market

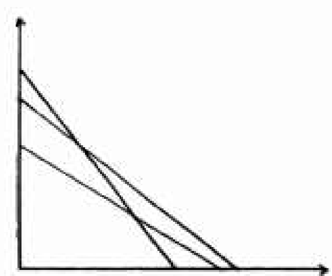

Fig. 2

situation means that every individual confronts completely separated markets from each other and the price of a commodity may be quite different from one individual to another. Such an impossibility theorem may be easily or directly suggested by Arrow's theorem through the analogy to the relation between the existence of the utility index and the strong axiom of revealed preference.[2/] This is why we say that Arrow's theorem has a close relationship to the economic choice under a certain market situation.

Now, if the market situation is different from that of complete price differential, Arrow's theorem cannot suggest directly the possibility of a non-trivial welfare function. Our main purpose in this paper is to prove an impossibility theorem of the welfare function where <u>every individual confronts the same market</u>. And this impossibility theorem may cover the impossibility theorem stated above. In fact, if the impossibility theorem is valid for the more restricted domain of definition of choice function, it is also valid for the less restricted domain.

To get the impossibility theorem, we assume that the individual preference orderings can change freely in a certain sense. We have impossibility theorems for both above-mentioned market situations under such an assumption. There arises no difference. But, if the individual preference orderings are restricted to those of constant marginal utility of money, the difference due to market situations arises. This difference may serve as an example of the importance of the introduction of the market situation into the social choice problem. Summarizing, the purpose of this paper is to study how the impossibility theorem of the social welfare function works, if the choice function is restricted to the <u>demand function</u>.

2. We shall use almost the same notation as in [7]. Let the number of individuals and commodities be m and n, respectively. We assume that the individual preference ordering is individualistic. The preference ordering of each individual is expressed by the functions of marginal rates of substitution. Let the marginal rate of substitution of the i-th individual between n-th and j-th commodities be

$$-\frac{\partial x_n^i}{\partial x_j^i} = s_j^i(x^i) .$$

Here, $x^i \equiv (x_1^i, \ldots , x_n^i)$ is a point in the nonnegative orthant of an n-dimensional space, and each component expresses the amount of the commodity consumed by the i-th individual. We assume that $s_j^i(x^i)$ $(j=1, \ldots , n-1)$ satisfy the integrability condition, i.e.,

$$\bar{s}_{jk}^i \equiv \frac{\partial s_j^i}{\partial x_k^i} - s_k^i \frac{\partial s_j^i}{\partial x_n^i} = \frac{\partial s_k^i}{\partial x_j^i} - s_j^i \frac{\partial s_k^i}{\partial x_n^i} \equiv \bar{s}_{kj}^i , \qquad (j \neq k) .$$

This assumption assures the consistency of the individual preference ordering. Further, we assume that matrix

$$(\bar{s}_{jk}^i) \equiv \bar{\sum}^i$$

is negative definite. The integrability condition is expressed by the symmetry of matrix $\bar{\sum}^i$. Let

$$x \equiv (x^1, \ldots , x^m) .$$

Vector x expresses a social state. A social state is expressed as a point in the nonnegative orthant of an $m \times n$-dimensional space. The preference ordering of the society among the social states is defined by the functions of the marginal rates of substitution between the n-th commodity consumed by the m-th individual and the j-th commodity consumed by the i-th individual. Now, the welfare function is a rule by which a preference ordering of the society is constructed from the preference orderings of individuals. In our case, the preference orderings of individuals or the society are expressed by the functions of marginal rates of substitution. We, therefore, can define the social welfare function as follows: We call a set of functions $\{\omega^i(x);\ i=1, \ldots ,m-1\}$ a welfare function, by which the marginal rates of substitution of society are constructed from the functions of the marginal rates of substitutions of individuals in the following way:

$$-\frac{\partial x_n^m}{\partial x_j^i} = \omega^i(x)\ s_j^i(x^i) \ .$$

The left-hand side expresses the marginal rate of substitution of the society between the m-th individual's n-th commodity and the i-th individual's j-th commodity. $\omega^i(x)$ expresses the marginal rate of substitution of the society between the n-th commodities consumed by m-th and i-th individuals; i.e.,

$$-\frac{\partial x_n^m}{\partial x_n^i} = \omega^i(x) \ .$$

This shows that if we know the marginal rates of substitution of the society between the n-th commodity consumed by m-th and that by any other individual, we can calculate the marginal rates of substitution between all commodities consumed by all individuals. As is easily seen, the marginal rate of substitution of the society between the n-th and j-th commodities consumed by an individual is equal to that of the individual. We defined the welfare function in a somewhat untraditional way. But, as was shown in [7], the welfare function of the Bergson-Samuelson type can be defined in this way.

In the above arguments, the preference orderings of individuals are assumed to be fixed, and thus $\omega^{\ell}(x)$ $(\ell=1, \ldots, m-1)$ are also assumed to be fixed. But, if the tastes of individuals change, what happens? If $\omega^{\ell}(x)$ $(\ell=1, \ldots, m-1)$ are fixed for any changing individual orderings, functions of the marginal rates of the society defined through $\omega^{\ell}(x)$ from $s_j^i(x^i)$ may not satisfy the integrability condition. Or, the consistency of the preference ordering of the society may be violated. In general, if function forms of $\omega^{\ell}(x)$ $(\ell=1, \ldots, m-1)$ can change when functions $s_j^i(x)$ change, or $\omega^{\ell}(x)$ are functionals of $s_j^i(x)$, we can find some $\omega^{\ell}(x)$ such that the functions of the marginal rates of the society defined through $\omega^{(\ell)}(x)$ from $s_j^i(x^i)$ satisfy the integrability condition. Then we have a problem. Can we find some non-functional welfare function? The meaning of the non-functional welfare function and its relation to Arrow's condition 3 were discussed in [7]. The non-functional welfare function is written as follows:

$$\omega^{\ell}(x) = \omega^{\ell}(s^1(x^1), \ldots, s^m(x^m), x) \qquad (\ell=1, \ldots, m-1) .$$

It is shown in [7] that such a welfare function should be trivial, i.e., dictatorial for any possible individual orderings.

Now, we know the relationship between the consistency of the preference ordering and the strong axiom of revealed preference. The latter is related to economic choice under budget constraint. Our social welfare function concerns the consistency of preference ordering of the society.

Analogously to the case of the demand function of a consumer, we may be able to relate the consistency of preference ordering of the society to the property of the demand function of the society. Before moving on to this problem, we reconsider the meaning of functions $\omega^{\ell}(x)$.

If $\omega^{\ell}(x) > 1$, this means that the loss of "welfare" due to the decrease of one marginal unit of the n-th commodity consumed by the m-th individual is smaller than the gain of "welfare" due to the increase of one marginal unit of the n-th commodity consumed by the ℓ-th individual. Thus, the transfer of one marginal unit of the n-th commodity from the m-th individual to the ℓ-th individual increases the "social welfare." If $\omega^{\ell}(x) < 1$, the inverse is true. Thus, the optimal distribution of commodities is achieved when

$$\omega^{\ell}(x) = 1 \quad \text{for all} \quad \ell ,$$

aside from neglecting the case of corner optima.

We shall consider the following socially optimal distribution of the total income to the individuals. Here, each individual confronts a market, separated completely from other markets. Let the price of the j-th commodity in the i-th market be p_j^i, and total income be I.

$$(1.1) \qquad \omega^{\ell}(s^1(x^1), \ldots , s^m(x^m), x) = 1 , \qquad (\ell=1, \ldots , m-1)$$

$$(1.2) \qquad \pi_j^i \equiv p_j^i / p_n^i = s_j^i(x^i) \;, \qquad (i=1, \ldots, m; \quad j=1, \ldots, n-1)$$

$$(1.3) \qquad \sum_{j=1}^{n} \sum_{i=1}^{m} p_j^i \, x_j^i = I \;.$$

Solutions x_j^i $(i=1, \ldots, m; \; j=1, \ldots, n)$ can be considered as functions of p_j^i and I, or explicitly

$$x_j^i(p^1, \ldots, p^m, I) \;.$$

Here, $p^i = (p_1^i, \ldots, p_n^i)$.

Now, function $x_j^i(p^1, \ldots, p^m, I)$ is a demand function for the j-th commodity of the i-th individual. The same commodity may have different prices in different markets. Hence, the same commodities consumed by different individuals should be treated as different commodities. We exclude in this paper the case of the corner optimum from (1.1)-(1.3). Thus, system (1.1)-(1.3) may not have a solution for some forms of the function $s_j^i(x^i)$ and some values of p_j^i and I. In an extreme case, (1.1)-(1.3) may have no solution for any forms of the function $s_j^i(x^i)$ and any values of p_j^i and I. In such a case, we may say that the demand functions defined by (1.1)-(1.3) satisfy the integrability condition in a trivial sense. For, the definition domains of the demand functions are empty. If we permit such a case, we may have a welfare function $\{\omega^\ell(s^1(x^1), \ldots, s^m(x^m), x); \; \ell=1, \ldots, m-1\}$ such that system (1.1)-(1.3) gives the demand functions satisfying the strong axiom of revealed preference. But, we confine ourselves to the case where some solutions of non-corner optima exist for some function forms of $s_j^i(x^i)$ and some values of p_j^i and I. Then, there is no welfare function such that (1.1)-(1.3) gives the demand function satisfying the strong axiom of revealed preference for any possible individual orderings. This can be seen by a more general theorem shown in the following sections. As was already stated, Arrow's theorem or the theorem in [7] concerns the choice function where the domain of definition is not confined to budget constraint sets. But, if the domain of definition is confined to budget constraint sets under the market situation of complete price differential, we have the impossibility theorem, too. The latter theorem is different from Arrow's or the theorem in [7], but it is directly suggested by them, through the analogy to the relation between the consistency of the preference ordering and the strong axiom of revealed preference of the demand functions. This is why we say that Arrow's theorem has a close relationship to the economic choice under a certain market situation.

3. In the previous section, we stated that Arrow's theorem suggests directly the impossibility theorem when the choice function is restricted to the demand functions under the market situation of complete price differential. But, such a market situation may be peculiar, and if the market situation is different, how does the impossibility theorem work? More specifically, if every individual confronts the same price situation, how does the impossibility theorem work? We call this market situation a perfectly competitive market. That is, $p_j^1 = p_j^2 = \cdots = p_j^m$ $(j=1,\ldots,n)$ should always hold in system (1.1)-(1.3). Such a market situation may be another extreme case. But, as is easily seen, the domain of definition of the choice function becomes narrower in this case than the case of the market situation of complete price differential. Thus, if we can prove the impossibility theorem for this case, it is also valid for the other extreme case. If we consider the preference ordering of a family - the society of minimum size - it may be natural to assume that every individual confronts the same price situation. In fact, the problem of the consistency of the preference ordering of a family may be one of the most important cases of the social choice problem. Moreover, we can get the impossibility theorem for the following intermediate market situation from this theorem: All individuals are divided into some groups, and every member of the same group confronts the same market. Now, we shall formulate our problem.

(2.1) $$\omega^{\ell}(s^1(x^1), \ldots, s^m(x^m), x) = 1 , \qquad (\ell=1, \ldots, m-1)$$

(2.2) $$\pi_j \equiv p_j/p_n = s_j^i(x^i) , \qquad (i=1, \ldots, m; \; j=1, \ldots, n-1)$$

$$\sum_{j=1}^{n} \sum_{i=1}^{m} p_j x_j^i = I \tag{2.3}$$

$$\sum_{i=1}^{m} x_j^i = x_j \quad (j=1, \ldots, n) . \tag{2.4}$$

The solution of this problem expresses the optimal distribution of commodities among individuals subject to budget constraint under the perfectly competitive market situation. The total demand function for the j-th commodity is

$$x_j(p_1, \ldots, p_n, I) = \sum_{i=1}^{m} x_j^i(p_1, \ldots, p_n, I) \quad (j=1, \ldots, n).$$

In this case, the same commodity has the same price for every individual and should be treated as such. If $\omega^{\ell}(s^1(x^1), \ldots, s^m(x^m), x)$, $(\ell=1, \ldots, m-1)$ are given arbitrarily, demand functions $x_j(p_1,\ldots,p_n,I)$ $(j=1, \ldots, n)$ may not satisfy the strong axiom of revealed preference. Then, <u>is there any welfare function such that the above system gives the demand functions satisfying the strong axiom of revealed preference for any possible individual orderings</u>? Here, we see some conspicuous difference from Arrow's case. In Arrow's case, or in its economic version, the ordering relation of the society is defined in an $m \times n$-dimensional space, but in the above case, a social ordering relation is defined in an n-dimensional space. That is, the social ordering relation is defined only on the <u>aggregated</u> alternatives.

Now, we shall consider the meaning of system (2.1)-(2.4). If total demand functions $x_j(p_1, \ldots, p_n, I)$ satisfy the strong axiom of revealed preference, we have some index $\psi(x_1, \ldots, x_n)$ such that

$x_j(p_1, \ldots, p_n, I)$ are the solutions of the following maximizing problem:

$$\text{Max } \psi(x_1, \ldots, x_n)$$

subject to

$$\sum_{j=1}^{n} p_j x_j = I .$$

$\psi(x_1, \ldots, x_n)$ defines the social indifference curves. Thus, our problem can be stated in another way. Is there any non-functional welfare function such that the social indifference curves always exist for any possible individual preference orderings? As was shown in [8], the social indifference curves exist if a welfare function of the Bergson-Samuelson type is given. In this case, the tastes of individuals are fixed and thus the welfare function $\{\omega^\ell(x); \ell=1, \ldots, m-1\}$ is also fixed. In our case, the tastes of individuals change freely. Our purpose is to show that any fixed welfare function $\{\omega^\ell(x); \ell=1, \ldots, m-1\}$, or more generally, any non-functional welfare function $\{\omega^\ell(s^1(x^1), \ldots, s^m(x^m), x); \ell=1, \ldots, m-1\}$, will not give the social indifference curves for some preference orderings of individuals. This is another impossibility theorem for the social welfare function.

We may be able to interpret (2.1)-(2.4) in another way. From (2.2) and (2.4), we have the "contract" curve" in the "Edgeworth Box" (Fig. 3). A point on this curve represents a Pareto optimal distribution of total amounts of all commodities. (2.1) is

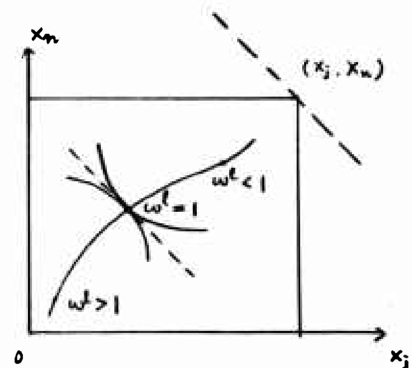

Fig. 3

the rule for picking up the most favorable point for the society on the "contract" curve. The slope of the indifference curves of individuals at this optimal point is $\pi_j = p_j/p_n$. The marginal rate of substitution of the society between n-th and j-th commodities at point $(x_1, \ldots, x_n)$ is equal to π_j. We may consider the inverse functions of the demand functions. Then, π_j can be considered as a function of $x_1, \ldots, x_n$. $\pi_j(x_1, \ldots, x_n)$ may or may not satisfy the integrability condition. Thus, our problem may be stated in this way: <u>Is there any non-functional welfare function such that</u> $\pi_j(x_1, \ldots, x_n)$ $(j=1, \ldots, n)$ <u>satisfy the integrability condition for any possible individual preference orderings</u>?

We shall state one more economic interpretation of our system: Let the i-th individual's demand function be

$$x^i_j(\pi_1, \ldots, \pi_{n-1}, M_i) \qquad (j=1, \ldots, n) .$$

Here, M_i is the income of the i-th individual in terms of the n-th commodity; that is, these functions are derived by solving the following system:

$$\pi_j = s^i_j(x^i) , \qquad j=1, \ldots, n-1$$

$$\sum_{j=1}^{n} \pi_j x^i_j = M_i \qquad (\pi_n = 1) .$$

This system shows the maximizing "utility" of the i-th individual under the budget constraint. If the function forms of $s^i_j(x^i)$ are given, the function forms of the demand functions are determined. Then, system (2.1)-(2.4) can be written as

$$(2.1)' \qquad \omega^{\ell}(\pi, \ldots, \pi, x^1(\pi, M_1), \ldots, x^m(\pi, M_m)) = 1, \; (\ell=1,\ldots,m-1)$$

$$(2.2)' \qquad \sum_{i=1}^{m} M_i = M(= I/p_n) ,$$

$$(2.3)' \qquad \sum_{i=1}^{m} x_j^i(\pi, M_i) = x_j \qquad (j=1, \dots , n) .$$

Here, π and M are parameters and $M_i (i=1, \dots , m)$ and $x_j (j=1, \dots ,n)$ are unknown. We introduced new unknown variables M_i and eliminated unknown variables x_j^i. The economic meaning of system (2.1)'-(2.3)' is easily explained; that is, the solution M_i of this system expresses the socially optimal distribution of <u>income</u> to the i-th individual. Thus, our welfare function ω^{ℓ} can be interpreted as the rule by which the optimal distribution of income among individuals is determined, if the demand structures of all individuals is known. We shall discuss the economic interpretation of our problem in more detail below.

We often used the term "any possible individual preference orderings." Here, we must explain the meaning of this term. We shall define the domain of the individual preference orderings as follows: $s_j^i(x^i)$ can take any value in an open interval (N_j, L_j) such that $0 < N_j$ and $L_j < \infty$. One restriction is that matrix $\sum^i$ is symmetric and negative definite. Otherwise the elements of $\bar{\sum}^i$ can change freely for fixed values of x^j and $s_j^i(x^i)$. We don't limit the value of $\frac{\partial s_j^i}{\partial x_k^i}$, provided $\bar{\sum}^i$ is symmetric and negative definite. From this freedom, we have the following two properties which will be used below. One is that every element of $\bar{\sum}^i$,

$$\frac{\partial s_j^i}{\partial x_k^i} - s_k^i \frac{\partial s_j^i}{\partial x_n^i} = \bar{s}_{jk}^i , \qquad (j \geq k) ,$$

can change freely in a certain interval. Another is that $\frac{\partial s_j^i}{\partial x_n^i}$ can also change freely in a certain interval. For

$$\frac{\partial s_j^i}{\partial x_k^i} - s_k^i \frac{\partial s_j^i}{\partial x_n^i} = \bar{s}_{jk}^i$$

of $\bar{\sum}^i$ can keep the same value by adjusting the value of $\frac{\partial s_j^i}{\partial x_k^i}$ even if $\frac{\partial s_j^i}{\partial x_n^i}$ changes.

We may say that the freedom of taste of individuals is expressed in two ways. One is the freedom to select the value of the marginal rate of substitution, and another is the freedom to select the value of the change rate of the marginal rate of substitution.

For the explanation of the meaning of this freedom, we shall cite two classes of preference orderings which do not satisfy the above-mentioned conditions.

One is the class of the constant marginal utility of money. We have no cardinal utility, so we must interpret the constancy of the marginal utility of money in some way. We follow the method of Hicks [5]. As was shown in [7],

$$\frac{\partial s_j^i}{\partial x_n^i} = 0 .$$

Thus, in this class of preference orderings, the value of $\frac{\partial s_j^i}{\partial x_n^i}$ has no freedom to change.

Another class is that of homogeneous preference orderings. In this case

$$\sum_{k=1}^{n} x_k^i \frac{\partial s_j^i}{\partial x_k^i} = 0 .$$

Thus, in this class, $\dfrac{\partial s_j^i}{\partial x_n^i}$ can take any value in a certain interval, but, at the same time, the elements of $\bar{\sum}^i$ should change, except in the case where x_j^i and s_j^i have some special relations.

As is easily seen, matrix $\bar{\sum}^i$ expresses the curvatures of the indifference surface at a point. The above mentioned freedom may be stated as follows. The slope of the indifference surface at a point can take any values in a certain interval (Fig. 4), the curvature at the point can take any values for a fixed slope, provided the indifference curve is convex (Fig. 5), and the change rate of the slope in one direction can take any values in a certain limit for a fixed slope, independently of the value of the curvature (Fig. 6).

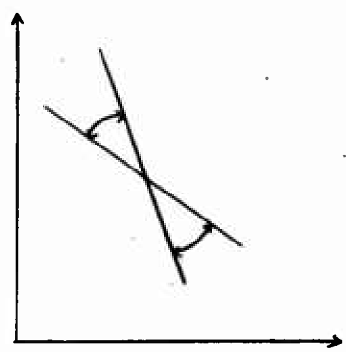

Fig. 4

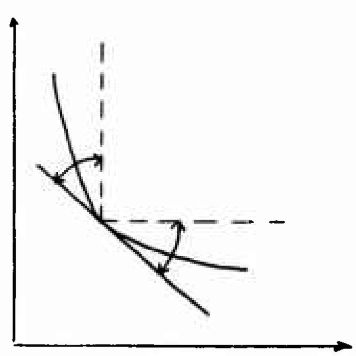

Fig. 5

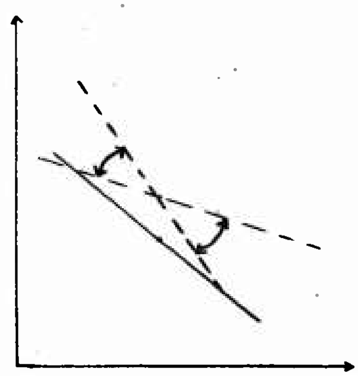

Fig. 6

4. We are concerned with non-functional welfare function $\left\{\omega^{\ell}(s^1(x^1),\ldots,s^m(x^m),x)\ ;\ell=1,\ldots,m-1\right\}$.[3] But in system (2.1)-(2-4), we always have

$$\pi_j = s_j^1(x^1) = \ldots = s_j^m(x^m), \quad (j=1,\ldots,n-1)\ ;$$

that is, our welfare function needs to be defined only for such values of $s_j^i(x^i)$. Thus, we may consider only the existence of functions of the following type:

$$\omega^{\ell}(\pi,x), \qquad (\ell=1,\ldots,m-1)\ .$$

System (2.1)-(2.4) can be written as follows:

$$(3.1) \qquad \omega^{\ell}(\pi,x) = 1, \quad (\ell=1,\ldots,m-1)\ ,$$

$$(3.2) \qquad \pi_j = s_j^i(x^i), \quad (i=1,\ldots,m;\ j=1,\ldots,n-1)\ ,$$

$$(3.3) \qquad \sum_{j=1}^{n} \sum_{i=1}^{m} \pi_j x_j^i = M(= I/p_n), \quad (\pi_n = 1)\ ,$$

$$(3.4) \qquad \sum_{i=1}^{m} x_j^i = x_j \quad (j=1,\ldots,n)\ .$$

As is easily seen, x_j $(j=1,\ldots,n)$ are functions of π and M. Here, we assume the differentiability of $\omega^i(\pi,x)$ with respect to every variable. Differentiating (3.1)-(3.4) with respect to π_k and M, we get

$$(4.1) \qquad \frac{\partial\omega^{\ell}}{\partial\pi_k} + \sum_{j=1}^{n} \sum_{i=1}^{m} \frac{\partial\omega^{\ell}}{\partial x_j^i} \frac{\partial x_j^i}{\partial\pi_k} = 0\ ,$$

$$(4.2) \qquad \delta_{jk} = \sum_{t=1}^{n} \frac{\partial s_j^i}{\partial x_t^i} \frac{\partial x_t^i}{\partial\pi_k}\ , \quad \delta_{jk} = 1 \quad \text{if } j = k$$
$$= 0 \quad \text{if } j \neq k\ ,$$

$$(4.3) \qquad \sum_{i=1}^{m} x_k^i + \sum_{i=1}^{m} \sum_{j=1}^{n} \pi_j \frac{\partial x_j^i}{\partial\pi_k} = 0\ ,$$

(4.4) $$\sum_{i=1}^{m} \frac{\partial x_j^i}{\partial \pi_k} = \frac{\partial x_j}{\partial \pi_k} ,$$

(5.1) $$\sum_{j=1}^{n} \sum_{i=1}^{m} \frac{\partial \omega^\ell}{\partial x_j^i} \frac{\partial x_j^i}{\partial M} = 0 ,$$

(5.2) $$0 = \sum_{t=1}^{n} \frac{\partial s_j^i}{\partial x_t^i} \frac{\partial x_t^i}{\partial M} ,$$

(5.3) $$\sum_{i=1}^{m} \sum_{j=1}^{n} \pi_j \frac{\partial x_j^i}{\partial M} = 1 ,$$

(5.4) $$\sum_{i=1}^{m} \frac{\partial x_j^i}{\partial M} = \frac{\partial x_j}{\partial M} .$$

From (4.i) and (5.i), (i = 1,2,3,4),

(6.1) $$\frac{\partial \omega^\ell}{\partial \pi_k} + \sum_{j=1}^{n} \sum_{i=1}^{m} \frac{\partial \omega^\ell}{\partial x_j^i} \left(\frac{\partial x_j^i}{\partial \pi_k} + x_k \frac{\partial x_j^i}{\partial M}\right) = 0 ,$$

(6.2) $$\delta_{jk} = \sum_{t=1}^{n} \frac{\partial s_j^i}{\partial x_t^i}\left(\frac{\partial x_t^i}{\partial \pi_k} + x_k \frac{\partial x_t^i}{\partial M}\right) ,$$

(6.3) $$0 = \sum_{i=1}^{m} \sum_{j=1}^{n} \pi_j\left(\frac{\partial x_j^i}{\partial \pi_k} + x_k \frac{\partial x_j^i}{\partial M}\right) ,$$

(6.4) $$\frac{\partial x_j}{\partial \pi_k} + x_k \frac{\partial x_j}{\partial \pi_k} = \sum_{i=1}^{m}\left(\frac{\partial x_j^i}{\partial \pi_k} + x_k \frac{\partial x_j^i}{\partial M}\right) .$$

Or, in matrix form,

$$[A]\begin{bmatrix}\bar{x}\\ \frac{\partial x}{\partial \pi}\end{bmatrix} \equiv \begin{bmatrix} \Sigma^1 & 0 & \dots & 0 & \sigma_n^1 & 0 & \dots & 0 & 0 \\ 0 & \Sigma^2 & \dots & 0 & 0 & \sigma_n^2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \Sigma^m & 0 & 0 & \dots & \sigma_n^m & 0 \\ W^1 & W^2 & \dots & W^m & w_n^1 & w_n^2 & \dots & w_n^m & 0 \\ \pi & \pi & \dots & \pi & 1 & 1 & \dots & 1 & 0 \\ I & I & \dots & I & 0 & 0 & \dots & 0 & -I \end{bmatrix} \begin{bmatrix}\bar{x}\\ \frac{\partial x}{\partial \pi}\end{bmatrix} = \begin{bmatrix} I \\ I \\ \vdots \\ I \\ \frac{\partial w}{\partial \pi} \\ 0 \\ 0 \end{bmatrix}$$

Here,

$$\Sigma^i \equiv \left(\frac{\partial s_j^i}{\partial x_k^i}\right), \quad \sigma_n^i \equiv \left(\frac{\partial s_j^i}{\partial x_n^i}\right), \quad \frac{\partial x}{\partial \pi} \equiv \left(\frac{\partial x_j}{\partial \pi_k} + x_k \frac{\partial x_j}{\partial M}\right),$$

$$W^i \equiv \left(\frac{\partial \omega^\ell}{\partial x_j^i}\right), \quad w_n^i \equiv \left(\frac{\partial \omega^\ell}{\partial x_n^i}\right), \quad \frac{\partial w}{\partial \pi} = \left(\frac{\partial \omega^i}{\partial \pi_j}\right),$$

and

$$\bar{x} = \begin{bmatrix} \bar{x}_{11}^1 & \dots & \bar{x}_{1n-1}^1 \\ \vdots & & \\ \bar{x}_{n-11}^1 & \dots & \bar{x}_{n-1n-1}^1 \\ \vdots & & \\ \bar{x}_{11}^m & \dots & \bar{x}_{1n-1}^m \\ \vdots & & \\ \bar{x}_{n-11}^m & \dots & \bar{x}_{n-1n-1}^m \\ \bar{x}_{n1}^1 & \dots & \bar{x}_{nn-1}^1 \\ \vdots & & \\ \bar{x}_{n1}^m & \dots & \bar{x}_{nn-1}^m \end{bmatrix}$$

Here, $\bar{x}_{jk}^i = \frac{\partial x_j^i}{\partial \pi_k} + x_k \frac{\partial x_j^i}{\partial M}$.

Now, $\frac{\partial x}{\partial \pi}$ should be symmetric and negative-definite for $x_j(\pi_1, \ldots, \pi_{n-1}, M)$ to satisfy the strong axiom of revealed preference. From this,

$$|A| \neq 0$$

and

$$(7) \qquad \begin{vmatrix} \Sigma^1 & & 0 & \sigma_n^1 & & 0 & \ell_j \\ & \ddots & & & \ddots & & \vdots \\ 0 & & \Sigma^m & 0 & & \sigma_n^m & \ell_j \\ W^1 & \cdots & W^m & w_n^1 & \cdots & w_n^m & \frac{\partial w}{\partial \pi_j} \\ \pi & \cdots & \pi & 1 & \cdots & 1 & 0 \\ \ell_k' & \cdots & \ell_k' & 0 & \cdots & 0 & 0 \end{vmatrix} = \begin{vmatrix} \Sigma^1 & & 0 & \sigma_n^1 & & 0 & \ell_k \\ & \ddots & & & \ddots & & \vdots \\ 0 & & \Sigma^m & 0 & & \sigma_n^m & \ell_k \\ W^1 & \cdots & W^m & w_n^1 & \cdots & w_n^m & \frac{\partial w}{\partial \pi_k} \\ \pi & \cdots & \pi & 1 & \cdots & 1 & 0 \\ \ell_j' & \cdots & \ell_j' & 0 & \cdots & 0 & 0 \end{vmatrix}$$

Here, ℓ_j is the j-th column of the unit matrix, and $\frac{\partial w}{\partial \pi_j}$ is the j-th column of matrix $\frac{\partial w}{\partial \pi}$.

After some manipulations on the determinants on both sides of (7),

$$(8) \qquad \begin{vmatrix} \bar{\Sigma}^1 & & 0 & \ell_j & \sigma_n^1 & & 0 \\ & \ddots & & \vdots & & \ddots & \\ 0 & & \bar{\Sigma}^m & \ell_j & 0 & & \sigma_n^m \\ \ell_k' & \cdots & \ell_k' & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \\ \bar{W}^1 & \cdots & \bar{W}^m & \frac{\partial w}{\partial \pi_j} & w_n^1 & \cdots & \partial w_n^m \end{vmatrix} = \begin{vmatrix} \bar{\Sigma}^1 & & 0 & \ell_k & \sigma_n^1 & & 0 \\ & \ddots & & \vdots & & \ddots & \\ 0 & & \bar{\Sigma}^m & \ell_k & 0 & & \sigma_n^m \\ \ell_j' & \cdots & \ell_j' & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \\ \bar{W}^1 & \cdots & \bar{W}^m & \frac{\partial w}{\partial \pi_k} & w_n^1 & \cdots & w_n^m \end{vmatrix}$$

Here,

$$\bar{\Sigma}^i = \Sigma^i - \sigma_n^i \pi \quad \text{and} \quad \bar{W}^i = W^i - w_n^i \pi \quad .$$

As $\pi_j = s_j^i$,

$$\bar{\Sigma}^i = \left(\frac{\partial s_j^i}{\partial x_k^i} - s_k^i \frac{\partial s_j^i}{\partial x_n^i}\right) = (\bar{s}_{jk}^i) \quad .$$

$\bar{\Sigma}^i$ is symmetric and negative-definite.

If $\bar{W}^i = 0$ and $\frac{\partial w}{\partial \pi} = 0$, (8) reduces to

$$\begin{vmatrix} \bar{\Sigma}^1 & & 0 & \ell_j \\ & \ddots & & \vdots \\ 0 & & \bar{\Sigma}^m & \ell_j \\ \ell_k' & \cdots & \ell_k' & 0 \end{vmatrix} \begin{vmatrix} 1 & \cdots & 1 \\ w_n^1 & \cdots & w_n^m \end{vmatrix} = \begin{vmatrix} \bar{\Sigma}^1 & & 0 & \ell_k \\ & \ddots & & \vdots \\ 0 & & \bar{\Sigma}^m & \ell_k \\ \ell_j' & \cdots & \ell_j' & 0 \end{vmatrix} \begin{vmatrix} 1 & \cdots & 1 \\ w_n^1 & \cdots & w_n^m \end{vmatrix}$$

As $\bar{\Sigma}^i$ is symmetric, this relation always holds. Our target is to show that the converse is also true for relation (8) to hold for any possible individual preference orderings.

We consider each side of (8) as a polynomial of the elements $\bar{s}^i_{jk}$ $(j \geqq k)$ of $\bar{\Sigma}^i$ and the elements of σ^i_n. As was mentioned, these elements can change freely in a certain interval without violating the symmetry and the negative definiteness of matrix $\bar{\Sigma}^i$. Then relation (8) should hold as an identity. This means that the coefficients of the same terms on both sides should be equal.

Consider the coefficients of $\sigma^1_{n1}, \bar{s}^1_{22} \cdots \bar{s}^1_{n-1n-1}\ \bar{s}^2_{11} \cdots \bar{s}^\ell_{n-1n-1} \cdots \bar{s}^m_{11} \cdots \bar{s}^m_{n-1n-1}$ on both sides of (8) for $j = 1$ and $k = 2$. Here, σ^i_{n1} is the first element of σ^i_n and $\bar{s}^i_{jj}$ is the j-th diagonal element of $\bar{\Sigma}^i$.

To calculate the coefficients, delete the row and column where each element of $\sigma^1_{n1}, \bar{s}^1_{22}, \ldots$ is located, and put the other elements of σ^1_n and $\bar{\Sigma}^i$ equal to zero. Then

$$0 = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 \\ \bar{W}^1_1 & \frac{\partial w}{\partial \pi_1} & w_n^2 & \cdots & w_n^m \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 \\ \bar{W}^1_1 & \frac{\partial w}{\partial \pi_2} & w_n^2 & \cdots & w_n^m \end{vmatrix} = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ \frac{\partial w}{\partial \pi_2} & w_n^1 & \cdots & w_n^m \end{vmatrix}$$

Here, $\bar{W}^1_1$ is the first column of $\bar{W}^1$.

Next, consider the coefficients of $\sigma^1_{n1}\ \bar{s}^1_{33} \cdots \bar{s}^1_{n-1n-1} \cdots \bar{s}^m_{11} \cdots \bar{s}^m_{n-1n-1}$.

$$0 = \begin{vmatrix} 0 & 0 & 0 & 0\ldots 0 \\ 0 & 1 & 0 & 0\ldots 0 \\ 0 & 0 & 0 & 1\ldots 1 \\ \bar{W}_1^1 & \bar{W}_2^1 & \frac{\partial w}{\partial \pi_1} & w_n^2\ldots w_n^m \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0\ldots 0 \\ 1 & 0 & 0 & 0\ldots 0 \\ 0 & 0 & 0 & 1\ldots 1 \\ \bar{W}_1^1 & \bar{W}_2^1 & \frac{\partial w}{\partial \pi_2} & w_n^2\ldots w_n^m \end{vmatrix} = \begin{vmatrix} 0 & 1\ldots 1 \\ \bar{W}_2^1 & w_n^2\ldots w_n^m \end{vmatrix}$$

By similar reasoning, we get that vector $\begin{pmatrix} 0 \\ \frac{\partial w}{\partial \pi_j} \end{pmatrix}$ or $\begin{pmatrix} 0 \\ \bar{W}_j^1 \end{pmatrix}$ is linearly dependent on $m - 1$ columns of $\begin{pmatrix} 1 \ldots 1 \\ w_n^1 \ldots w_n^m \end{pmatrix}$ if $\begin{pmatrix} 1 \\ w_n^j \end{pmatrix}$ is included in those $m - 1$ columns.

Next, consider the coefficients of $\bar{s}_{11}^1\ \sigma_{n2}^1\ \bar{s}_{33}^1 \ldots \bar{s}_{n-1n-1}^1 \ldots \bar{s}_{11}^m \ldots \bar{s}_{n-1n-1}^m$.

$$\begin{vmatrix} 0 & 1\ldots 1 \\ \frac{\partial w}{\partial \pi} & w_n^2\ldots w_n^m \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0\ldots 0 \\ 0 & 0 & 1\ldots 1 \\ \bar{W}_2^1 & \frac{\partial w}{\partial \pi_1} & w_n^2\ldots w_n^m \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0\ldots 0 \\ 0 & 0 & 1\ldots 1 \\ \bar{W}_2^1 & \frac{\partial w}{\partial \pi_2} & w_n^2\ldots w_n^m \end{vmatrix} = 0 .$$

Consider the coefficients of $\sigma_{n2}^1\ \bar{s}_{33}^1 \ldots \bar{s}_{n-1n-1}^1 \ldots \bar{s}_{11}^m \ldots \bar{s}_{n-1n-1}^m$.

$$\begin{vmatrix} 0 & 1\ldots 1 \\ \bar{W}_1^1 & w_n^2\ldots w_n^m \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0\ldots 0 \\ 0 & 1 & 0 & 0\ldots 0 \\ 0 & 0 & 0 & 1\ldots 1 \\ \bar{W}_1^1 & \bar{W}_2^1 & \frac{\partial w}{\partial \pi_1} & w_n^2\ldots w_n^m \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0\ldots 0 \\ 1 & 0 & 0 & 0\ldots 0 \\ 0 & 0 & 0 & 1\ldots 1 \\ \bar{W}_1^1 & \bar{W}_2^1 & \frac{\partial w}{\partial \pi_2} & w_n^2\ldots w_n^m \end{vmatrix} = 0.$$

By similar reasoning, we get that vector $\begin{pmatrix} 0 \\ \frac{\partial w}{\partial \pi_j} \end{pmatrix}$ or $\begin{pmatrix} 0 \\ \bar{W}_j^1 \end{pmatrix}$ is linearly dependent on $m - 1$ columns of $\begin{pmatrix} 1 & \ldots 1 \\ w_n^1 & \ldots w_n^m \end{pmatrix}$ if $\begin{matrix} 1 \\ w_n^j \end{matrix}$ is not included in those $m - 1$ columns. Thus, we get that $\begin{pmatrix} 0 \\ \frac{\partial w}{\partial \pi_j} \end{pmatrix}$ or $\begin{pmatrix} 0 \\ \bar{W}_j^1 \end{pmatrix}$ is linearly dependent on any $m - 1$ columns of $\begin{pmatrix} 1 & \ldots & 1 \\ w_n^1 & \ldots & w_n^m \end{pmatrix}$.

If $\begin{pmatrix} 1 \\ w_n^1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ w_n^m \end{pmatrix}$ are linearly dependent, every $m \times m$ determinant constructed by m vectors of $\begin{pmatrix} 0 \\ \overline{W}_j^i \end{pmatrix}$, $\begin{pmatrix} 0 \\ \frac{\partial w}{\partial \pi_j} \end{pmatrix}$ and $\begin{pmatrix} 1_j \\ w_n \end{pmatrix}$ should vanish.

Then

$$|A| = (-1)^n \begin{vmatrix} \overline{\Sigma}^1 & & 0 & \sigma_n^1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \overline{\Sigma}^m & 0 & & \sigma_n^m \\ 0 & \ldots & 0 & 1 & \ldots & 1 \\ \overline{W}^1 & \ldots & \overline{W}^m & w_n^1 & \ldots & w_n^m \end{vmatrix} = 0 .$$

For the Laplace expansion of this determinant, according to the last m rows, shows the result. Thus,

$$\begin{vmatrix} 1 & \ldots & 1 \\ w_n^1 & \ldots & w_n^m \end{vmatrix} \neq 0 .$$

Then,

$$(9) \qquad \begin{pmatrix} 0 \\ \overline{W}_j^i \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} 0 \\ \frac{\partial w}{\partial \pi_j} \end{pmatrix} = 0 \quad (i = 1,\ldots,m;\ j = 1,\ldots,n-1) .$$

This is the desired result. As is easily seen, these relations of derivatives of $\omega^i(\pi,x)$ should hold at the point where

$$\omega^\ell(\pi,x) = 1 \quad (\ell = 1,\ldots,m-1) .$$

Now if system (3.1)-(3.4) has a solution such that $x > 0$ for some values of π_j and M, and some function forms of $s_j^i(x^i)$, relation (9) should hold at point (π,x). From the differentiability of ω^ℓ with respect to π and x, we may assume that system (3.1)-(3.4) has a solution for any values of π belonging to a certain interval and some values of M and some function forms of $s_j^i(x^i)$.

Consider relations (3.1) and (3.3):

$$(3.3) \qquad \sum_i \sum_j \pi_j x_j^i = M ,$$

$$(3.1) \qquad \omega^{\ell}(\pi,x) = 1 .$$

From these relations, we can solve x_n^i $(i = 1,\ldots,m)$ as functions in other variables π_j and x_j^i $(j \neq n)$. As is easily seen, the Jacobian is

$$\begin{vmatrix} 1 & \ldots & 1 \\ w_n^1 & \ldots & w_n^m \end{vmatrix} \neq 0 .$$

Differentiating (3.3) and (3.1) with respect to x_j^k, we get

$$\pi_j + \sum_{i=1}^{m} \frac{\partial x_n^i}{\partial x_j^k} = 0 ,$$

$$\frac{\partial \omega^{\ell}}{\partial x_j^k} + \sum_{i=1}^{m} \frac{\partial \omega^{\ell}}{\partial x_n^i} \frac{\partial x_n^i}{\partial x_j^k} = 0 .$$

As

$$\frac{\partial \omega^{\ell}}{\partial x_j^k} - \pi_j \frac{\partial \omega^{\ell}}{\partial x_n^k} = 0 ,$$

we get

$$\pi_j + \sum_{i=1}^{m} \frac{\partial x_n^i}{\partial x_j^k} = 0,$$

$$\pi_j \frac{\partial \omega^{\ell}}{\partial x_n^k} + \sum_{i=1}^{m} \frac{\partial \omega^{\ell}}{\partial x_n^i} \frac{\partial x_n^i}{\partial x_j^k} = 0 .$$

Then

$$\begin{bmatrix} 1 & \cdots & 1 \\ \partial\omega^1/\partial x_n^1 & \cdots & \partial\omega^1/\partial x_n^m \\ \vdots & & \vdots \\ \partial\omega^{m-1}/\partial x_n^1 & \cdots & \partial\omega^{m-1}/\partial x_n^m \end{bmatrix} \begin{bmatrix} \partial x_n^1/\partial x_j^k \\ \vdots \\ \partial x_n^k/\partial x_j^k + \pi_j \\ \vdots \\ \partial x_n^m/\partial x_j^k \end{bmatrix} = 0 .$$

As

$$\begin{vmatrix} 1 & \cdots & 1 \\ \partial\omega^1/\partial x_n^1 & \cdots & \partial\omega^1/\partial x_n^m \\ \vdots & & \vdots \\ \partial\omega^{m-1}/\partial x_n^1 & \cdots & \partial\omega^{m-1}/\partial x_n^m \end{vmatrix} \equiv \begin{vmatrix} 1 & \cdots & 1 \\ w_n^1 & \cdots & w_n^m \end{vmatrix} \neq 0 ,$$

we get

$$\frac{\partial x_n^i}{\partial x_j^k} = 0 \quad , \quad (k \neq i)$$

$$\frac{\partial x_n^k}{\partial x_j^k} = - \pi_j \ .$$

From this, $\sum_{i=1}^{n} \pi_j x_j^i$ should be a constant which depends on π and M if x_j^i satisfy relations (3.1) and (3.3).

Put this constant $\phi_i(\pi,M)$ then

$$\sum_{j=1}^{n} \pi_j x_j^i = \phi_i(\pi,M)$$

or

$$\sum_{j=1}^{n} \pi_j x_j^i = \phi_i(\pi, \sum_i \sum_j \pi_j x_j^i) \ .$$

Thus, relations (3.1) and (3.3) are equivalent to $\sum_i \sum_j \pi_j x_j^i = M,$ and

$$\sum_j \pi_j x_j^{\ell} / \phi_{\ell}(\pi, \ \sum_i \sum_j \pi_j x_j^i) \equiv \overline{\omega}^{\ell}(\sum_j \pi_j x_j^1, \ldots, \sum_j \pi_j x_j^m) = 1 \ .$$

This means that if there exist a non-trivial and non-functional welfare function, then there should exist an equivalent welfare function such that whether some income distribution is optimal or not is judged only by the magnitude of every individual's income and prices of all commodities.

That is, such an income distribution rule does not need the demand structure of every individual in detail to determine the socially optimal distribution. Or, the function form of each $x_j^i(\pi, M_i)$ is not needed, but only the value of $\sum_{j=1}^{n} \pi_j x_j^i = M_i$ and π is needed.

Now consider the welfare function, $\bar{\omega}^\ell(\pi, x)$. Then, repeating the same procedure as for $\omega^\ell(\pi, x)$ we get

$$\frac{\partial \bar{\omega}^\ell}{\partial \pi_j} = 0 \quad \text{and} \quad \frac{\partial \bar{\omega}^\ell}{\partial x_j^i} - \pi_j \frac{\partial \bar{\omega}^\ell}{\partial x_n^i} = 0 \ ,$$

and

$$\begin{vmatrix} 1 & \cdots & 1 \\ \partial \bar{\omega}^1/\partial x_n^1 & \cdots & \partial \bar{\omega}^1/\partial x_n^m \\ \vdots & & \\ \partial \bar{\omega}^{m-1}/\partial x_n^1 & \cdots & \partial \bar{\omega}^{m-1}/\partial x_n^m \end{vmatrix} \neq 0 \ ,$$

at a point where

$$\bar{\omega}^\ell = 1 \quad \text{and} \quad \sum_i \sum_j \pi_j x_j^i = M \ .$$

Now

$$\frac{\partial \bar{\omega}^\ell}{\partial \pi_k} = \frac{1}{\phi_\ell} [x_k^\ell - (\phi_{\ell k} + \phi_{\ell M} \sum_{i=1}^{m} x_k^i)] = 0 \tag{10}$$

at a point where

$$\bar{\omega}^\ell = 1 \quad \text{and} \quad \sum_i \sum_j \pi_j x_j^i = M \ .$$

Here,

$$\frac{\partial \phi_\ell}{\partial \pi_k} \equiv \phi_{\ell k} \ , \quad \text{and} \quad \frac{\partial \phi_\ell}{\partial M} = \phi_{\ell M} \ .$$

From (10), we get

(11) $$x_k^{\ell} = \phi_{\ell k} + \phi_{\ell M}\left(\sum_{i=1}^{m} x_k^i\right), \qquad (\ell=1, \ldots, m-1).$$

Then

$$\sum_{i=1}^{m} x_k^i = \sum_{i=1}^{m-1} \phi_{ik} + x_k^m + \left(\sum_{i=1}^{m-1} \phi_{iM}\right)\left(\sum_{i=1}^{m} x_k^i\right).$$

Two cases should be considered.

(i) $$1 - \sum_{i=1}^{m-1} \phi_{iM} = 0.$$

(ii) $$1 - \sum_{i=1}^{m-1} \phi_{iM} \neq 0.$$

In case (i), we get

$$x_k^m = -\sum_{i=1}^{m-1} \phi_{ik}, \qquad (k=1, \ldots, n-1).$$

Here, ϕ_{ik} depends only on the values of π and M. This shows that the values of x_k^m $(k=1, \ldots, n-1)$ are determined independently from the taste of m-th individual. On the other hand, system

(12.1) $$M_{\ell} = \phi_{\ell}(\pi, M), \qquad \ell=1, \ldots, m-1$$

(12.2) $$M_m = M - \sum_{\ell=1}^{m-1} \phi_{\ell}(\pi, M)$$

expresses an income distribution rule and the values of $M_i(i=1,\ldots,m)$ are determined by this rule uniquely for given values of π and M.

The optimal commodity allocation to m-th individual is determined as the solution of the following system:

$$\sum_{j=1}^{n} \pi_j x_j^m = M_m ,$$

$$\pi_j = s_j^m(x^m) , \qquad j=1, \ldots , n-1 .$$

As is easily seen, the solution of this system depends on the function forms of $s_j^m(x^m)$. In fact, every set of positive values of x_j^m which satisfies the budget constraint can be the solution of this system for some suitable function forms of $s_j^m(x^m)$. For, from the freedom of individual preference orderings, the value of $s_j^m(x^m)$ at an arbitrary point can be taken equal to π_j. Thus, we reach a contradiction: According to this income distribution rule, the values of x_k^m should be determined independently of the preference of the m-th individual on the one hand, and the values of the same variables should be determined dependently of the preference of the m-th individual on the other hand.

In case (ii),

$$(13) \qquad \sum_{i=1}^{m} x_k^i = [\sum_{i=1}^{m-1} \phi_{ik} + x_k^m] / [1 - \sum_{i=1}^{m-1} \phi_{iM}] .$$

Then, from (11) and (13),

$$(14) \qquad x_k^\ell = \phi_{\ell k} + \phi_{\ell M}[\sum_{i=1}^{m-1} \phi_{ik} + x_k^m] / [1 - \sum_{i=1}^{m-1} \phi_{iM}]$$

$$\equiv \xi_k^\ell(x_k^m) \quad , \quad \ell = 1,\ldots,m-1; \ k = 1,\ldots,n-1.$$

Here, $\phi_{\ell k}$ and $\phi_{\ell M}$ depend only on the values of π and M . This shows that the value of x_k^ℓ has a unique linear relationship to the value of x_k^m .

Now, x_k^m should be an optimal commodity allocation to the m-th individual. Thus,

$$\sum_{j=1}^{n} \pi_j x_j^m = M_m \quad ,$$

$$\pi_j = s_j^m(x_j^m), \qquad j = 1,\ldots,n-1 \ .$$

Here, M_m is determined as the solution of system (12). The values of x_j^m depend on the function forms of $s_j^m(x_j^m)$. If x_j^m are determined, the values of x_k^ℓ are uniquely determined independently from the preference of the ℓ-th individual. On the other hand, the optimal commodity allocation to the ℓ-th individual should satisfy relations

$$\sum_{j=1}^{n} \pi_j x_j^\ell = M_\ell \quad ,$$

$$\pi_j = s_j^\ell(x^\ell), \quad j = 1,\ldots,n-1 \ .$$

The solution of this system depends on the function forms of $s_j^\ell(x^\ell)$. In this case, according to this income distribution rule, the optimal commodity allocation to the ℓ-th individual $(\ell \neq m)$ should be determined dependently by the preference of the m-th individual, but not dependently by his own preference on the one hand, and it should be determined dependently on his own preference on the other hand. Thus, we reach a contradiction. Therefore, $\bar{\omega}^\ell(\pi,x)$ cannot give the consistent demand functions for some function forms of $s_j^i(x^i)$. Hence, there exists no non-trivial and non-functional welfare function which gives consistent demand functions for any possible individual preference orderings.

We have confined ourselves to the case where the optimal social state is achieved at some internal point for some values of $s_j^i(x^i)$ and p_j and I. If we consider the case where the optimal social state is always achieved at some corner point for every possible value of $s_j^i(x^i)$ and

p_j and I, we can find some welfare function. In [7], we get only possible welfare functions for the individual case, such as

$$\omega^{\ell}(s^1(x^1),\ldots,s^m(x^m),x) = 0 \qquad (\ell = 1,\ldots,m-1) .$$

In this case,

$$\begin{vmatrix} 1 & \ldots & 1 \\ w_n^1 & \ldots & w_n^m \end{vmatrix} = 0 \quad \text{and} \quad A = 0 .$$

We have no solutions for system (2.1)-(2.4). In such a case, the optimal social state can be achieved if all commodities are totally allocated to the m-th individual. Thus, our impossibility theorem does not deny the existence of such a welfare function. The imposed or dictatorial welfare function in Arrow's theorem can be considered in such a way that the optimum social distribution is always achieved at the corner.

5. We can give another interpretation of our impossibility theorem. In Section 3 we gave an economic interpretation of our problem in terms of the optimal distribution of income among individuals. Our non-functional welfare function is interpreted as the rule of the determination of the income distribution.

$$\omega^{\ell}(\pi_1,\ldots,\pi,x^1(\pi,M_1),\ldots,x^m(\pi,M_m)) = 1 \quad (\ell = 1,\ldots,m-1)$$

expresses the rule; that is, if π and the total income M are given, the above equations determine the optimal income distribution. We stated that this rule determines the optimal income distribution if the demand structures of all individuals are given. But here we need not know the whole or complete structures of the demand functions of all individuals. That is, if we can know only the income-consumption functions $x^i(\pi,M_i)$ of all individuals under the prevailing price π, (2.1) and (2.2) can determine the optimal income distribution. This income distribution rule does not need the demand structures of individuals under other price constellations. Thus, our non-functional welfare function can be interpreted as the rule of income distribution if we only know the income-consumption functions of individuals under the prevailing price constellation. Our impossibility theorem assures that the total demand functions derived from such income distribution rule do not satisfy the strong axiom of revealed preference for some demand functions of individuals, if the demand functions of all individuals can change freely.

The freedom of tastes of individuals can be expressed in terms of demand functions. Corresponding to the freedom to select the marginal rate of substitution, the demand functions can take any values at any price constellations in the interval described in Section 3 and any positive values of income. Corresponding to the symmetry and the negative-definiteness of matrix $\bar{\bar{\sum}}_i$, matrix $[\frac{\partial x^i}{\partial \pi}]$ should be symmetric and negative-definite. Here,

$$[\frac{\partial x^i}{\partial \pi}] = [\frac{\partial x^i_j}{\partial \pi_k} + x^i_k \frac{\partial x^i_j}{\partial M}] \qquad (j = 1,\ldots,n-1;\ k = 1,\ldots,n-1).$$

In fact, $[\frac{\partial x^i}{\partial \pi}]$ is the inverse of $\bar{\sum}^i$.

As a special case for this theorem, we get the following corollary.

We consider the income distribution rule which does not take into account the demand structures of individuals. Such a rule can be expressed as follows:

$$\omega^{\ell}(\pi, M_1, \ldots, M_m) = 1, \quad (\ell = 1,\ldots,m-1).$$

This type of rule can be considered as a special case of the above-discussed rule,

$$\omega^{\ell}(\pi,\ldots,\pi,\ x^1(\pi,M_1),\ldots,x^m(\pi,M_m)) = 1 \quad (\ell = 1,\ldots,m-1).$$

As already shown in Section 4, such a distribution rule may violate the integrability condition for some individual preference orderings. In fact, we proved the general impossibility theorem by reducing the impossibility of this special distribution rule. Or, we may say in a converse way that even if the income distribution rule is generalized so as to take into account the income-consumption structure of every individual, the demand function of society derived from it may violate the strong axiom of revealed preference for some individual orderings.

The income distribution rule, which takes into account not only the income-consumption structure but also the price-consumption structure of every individual, is nothing but the functional welfare function. Of course, we can find an income distribution rule such that the demand functions derived from this rule satisfy the strong axiom of revealed preference for any individual orderings.

6. In this section we shall study the case where the marginal utility of money is constant. We shall interpret the constant marginal utility of money in Hicks's sense [5]. In this case, by a simple calculation,

$$\frac{\partial x}{\partial \pi} = (\bar{\sum}^1)^{-1} + \dots + (\bar{\sum}^m)^{-1} \quad .$$

$\bar{\sum}^i$ is symmetric and negative-definite. Hence, $\frac{\partial x}{\partial \pi}$ is also symmetric and negative-definite. Thus, any welfare function can give the demand function satisfying the strong axiom of revealed preference. It is shown in [7] that the only possible welfare function is

$$\omega^\ell(s^1(x^1),\dots,s^m(x^m),x) = k_\ell \; ; \; \ell = 1,\dots,m-1 \ ,$$

if the definition domain of the choice function consists of all subsets. As is easily seen, $|A| = 0$ for such a welfare function.

This difference is caused by the difference of the domains of definition of the choice functions.

Moreover, it is impossible to find a welfare function such that system (1.1)-(1.3) gives the demand functions satisfying the strong axiom of revealed preference, even if the individual preference orderings are restricted to those of constant marginal utility of money. The proof will be omitted.

Thus, we see that the impossibility theorem depends on the market situation assumed.

7. In this section, we shall consider the case of "homogeneous utility." We can find a non-functional welfare function such that the demand functions satisfy the strong axiom of revealed preference for any possible individual preference orderings of homogeneous utility. The rule of "a shibboleth," called so by Samuelson [8], or the income distribution rule of constant relative share, is such a welfare function. This can be seen as follows.

Consider the following welfare function:

$$\omega^{\ell} \equiv \alpha_{\ell} \sum_{j=1}^{n} \pi_j x_j^m \Big/ \left(\sum_{j=1}^{n} \pi_j x_j^{\ell}\right) \quad .$$

Here, the α_{ℓ}'s are non-negative constants.

Let

$$\beta_{\ell} \equiv \alpha_{\ell} \Big/ \left(\sum_{i=1}^{m-1} \alpha_i + 1\right) \geqq 0 \quad ,$$

$$\beta_m \equiv 1 \Big/ \left(\sum_{i=1}^{m-1} \alpha_i + 1\right) \geqq 0 \quad .$$

Then

$$\sum_{i=1}^{m} \beta_i = 1 \quad .$$

Put

$$\sum_{j=1}^{n} \pi_j x_j^{\ell} \equiv M_{\ell} \qquad (\ell = 1, \ldots, m) \quad .$$

Then $\omega^{\ell} = 1$ is equivalent to $M_{\ell} = \beta_{\ell} M$. This means that our welfare function expresses the rule of income distribution of constant relative share. In case of "homogeneous utility,"

$$x_j^{\ell}(\pi, M_{\ell}) = a_j^{\ell}(\pi) M_{\ell} \quad .$$

Thus,

$$\frac{\partial x_j^\ell}{\partial \pi_k} + x_k^\ell \frac{\partial x_j^\ell}{\partial M_\ell} = \left(\frac{\partial a_j^\ell}{\partial \pi_k} + a_k^\ell a_j^\ell\right) M_\ell \quad .$$

From this, matrix

$$A^\ell \equiv \left[\frac{\partial a_j^\ell}{\partial \pi_k} + a_k^\ell a_j^\ell\right]$$

is symmatric and negative-definite.

As is easily seen, matrix

$$\Psi \equiv \left[\sum_{\ell=1}^{m} \beta_\ell \left(\frac{\partial a_j^\ell}{\partial \pi_k} + a_k^\ell a_j^\ell\right)\right] \equiv \sum_{\ell=1}^{m} \beta_\ell A^\ell$$

is also symmetric and negative-definite.

Next, consider the total demand functions $x_j(\pi,M)$.

$$x_j(\pi,M) = \sum_{\ell=1}^{m} x_j^\ell(\pi,M_\ell) \qquad (M_\ell = \beta_\ell M)$$

$$= \sum_{\ell=1}^{m} \beta_\ell a_j^\ell M \quad .$$

From this,

$$\frac{\partial x_j}{\partial \pi_k} + x_k \frac{\partial x_j}{\partial M} = \left[\sum_{\ell=1}^{m} \beta_\ell \frac{\partial a_j^\ell}{\partial \pi_k} + \left(\sum_{\ell=1}^{m} \beta_\ell a_j^\ell\right)\left(\sum_{\ell=1}^{m} \beta_\ell a_j^\ell\right)\right] M \quad .$$

Our target is to show that matrix

$$\phi \equiv \left[\sum_{\ell=1}^{m} \beta_\ell \frac{\partial a_j^\ell}{\partial \pi_k} + \left(\sum_{\ell=1}^{m} \beta_\ell a_j^\ell\right)\left(\sum_{\ell=1}^{m} \beta_\ell a_j^\ell\right)\right]$$

is symmetric and negative-definite. Now, as

$$\frac{\partial a_j^\ell}{\partial \pi_k} + a_k^\ell a_j^\ell = \frac{\partial a_k^\ell}{\partial \pi_j} + a_j^\ell a_k^\ell \quad ,$$

$$\frac{\partial a_j^\ell}{\partial \pi_k} = \frac{\partial a_k^\ell}{\partial \pi_j} \quad .$$

From this, ϕ is symmetric.

$$\xi'\phi\xi = \sum_k \sum_j \sum_\ell \beta_\ell \frac{\partial a_j^\ell}{\partial \pi_k} \xi_j \xi_k + \sum_k \sum_j \left(\sum_\ell \beta_\ell a_j^\ell\right)\left(\sum_\ell \beta_\ell a_k^\ell\right) \xi_j \xi_k =$$

$$= \sum_k \sum_j \sum_\ell \beta_\ell \frac{\partial a_j^\ell}{\partial \pi_k} \xi_j \xi_k + \left\{\sum_\ell \beta_\ell \left(\sum_j a_j^\ell \xi_j\right)\right\}^2 .$$

Here,

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{n-1} \end{pmatrix}.$$

Now,

$$\left\{\sum_\ell \beta_\ell \left(\sum_j a_j^\ell \xi_j\right)\right\}^2 \leqq \sum_\ell \beta_\ell \left(\sum_j a_j^\ell \xi_j\right)^2 .$$

This can be seen as follows. The following inequality (Schwarz) is well-known:

$$\left(\sum_\ell \zeta_\ell \eta_\ell\right)^2 \leqq \left(\sum_\ell \zeta_\ell^2\right)\left(\sum_\ell \eta_\ell^2\right) .$$

Put

$$\zeta_\ell = \sqrt{\beta_\ell} \quad \text{and} \quad \eta_\ell = \sqrt{\beta_\ell} \sum_j a_j^\ell \xi_j .$$

Then

$$\left(\sum \zeta_\ell \eta_\ell\right)^2 = \left\{\sum_\ell \beta_\ell \left(\sum_j a_j^\ell \xi_j\right)\right\}^2$$

$$\sum_\ell \zeta_\ell^2 = \sum_\ell \beta_\ell = 1 \quad \text{and} \quad \sum_\ell \eta_\ell^2 = \sum_\ell \beta_\ell \left(\sum_j a_j^\ell \xi_j\right)^2 .$$

Thus,

$$\xi'\phi\xi \leqq \sum_k \sum_j \sum_\ell \beta_\ell \frac{\partial a_j^\ell}{\partial \pi_k} \xi_j \xi_k + \sum_\ell \beta_\ell \left(\sum_j a_j^\ell \xi_j\right)^2$$

$$= \sum_k \sum_j \sum_\ell \beta_\ell \frac{\partial a_j^\ell}{\partial \pi_k} \xi_j \xi_k + \sum_k \sum_j \sum_\ell \beta_\ell a_j^\ell a_k^\ell \xi_j \xi_k$$

$$= \xi'\Psi\xi < 0 \quad \text{for} \quad \xi \neq 0 .$$

This shows that ϕ is negative-definite.

The above result was obtained by Eisenberg [3] through another channel. Instead of a locally defined welfare function, the following Bergson-Samuelson type welfare function is treated in [3]:

$$\psi(x^1,\ldots,x^m) = \prod_{\ell=1}^{m} [u^\ell(x^\ell)]^{\beta_\ell} \quad (\sum_{\ell=1}^{m} \beta_\ell = 1) \quad .$$

Here, $u^\ell(x^\ell)$ expresses the homogeneous utility index of the i-th individual. But this welfare function reduces to a "shibboleth" income distribution rule. This can be seen as follows:

Put

$$\psi(x^1,\ldots,x^m) = C \ .$$

Then

$$\beta_\ell u_n^\ell/u^\ell + \beta_m \ u_n^m/u^m \ \frac{\partial x_n^m}{\partial x_n^\ell} = 0 \ .$$

Here,

$$u_n^\ell \equiv \frac{\partial u^\ell}{\partial x_n^i} \quad .$$

Thus,

$$-\frac{\partial x_n^m}{\partial x_n^\ell} \equiv \omega^\ell(x) = \frac{\beta_\ell u_n^\ell/u^\ell}{\beta_m \ u_n^m/u^m} \quad .$$

On the other hand, from the homogeneity of u^ℓ ,

$$u^\ell = \sum_{j=1}^{n} u_j^\ell \ x_j^\ell \quad .$$

Here,

$$u_j^\ell = \frac{\partial u^\ell}{\partial x_j^\ell} \quad .$$

Thus,

$$u^\ell/u_n^\ell = \sum_{j=1}^{n} u_j^\ell/u_n^\ell \ \ x_j^\ell \ =$$

$$= \sum_{j=1}^{n} s_j^{\ell}(x^{\ell}) x_j^{\ell} \quad .$$

For,

$$- \frac{\partial x_n^{\ell}}{\partial x_j^{\ell}} = s_j^{\ell}(x^{\ell}) = \frac{u_j^{\ell}}{u_n^{\ell}} \quad .$$

In our case,

$$\pi_j = s_j^{\ell}(x^{\ell}) \quad .$$

Hence,

$$u^{\ell}/u_n^{\ell} = \sum_{j=1}^{n} \pi_j \, x_j^{\ell} \quad .$$

Thus, the income distribution rule is

$$\omega^{\ell} = \alpha_{\ell} \sum_{j=1}^{n} \pi_j \, x_j^{m} \Big/ \left(\sum_{j=1}^{n} \pi_j \, x_j^{\ell} \right) = 1 \qquad \left(\alpha_{\ell} \equiv \frac{\beta_{\ell}}{\beta_m} \right) .$$

This is a "shibboleth income distribution rule."

It is shown in [8] that if a social welfare function of the Bergson-Samuelson type is given, there exist social indifference curves. As is easily seen, the theorem about the existence of the social utility index in [3] is only a special case of Samuelson's theorem in [8]. If the welfare function of special type (11) did not have some economically meaningful characters, the theorem in [3] might be trivial. But, as was shown above, welfare function (11) expresses the income distribution rule of constant relative share. As is mentioned in [8], it is generally incompatible with the maximization of a social welfare function. But if individual orderings are restricted to the homogeneous ones, a "shibboleth" income distribution rule is compatible with the maximization of social welfare.

It is shown in [1] that if individual preference orderings are restricted to the single peaked ones, the simple majority decision rule satisfies Arrow's five conditions. This theorem shows that if individual preference orderings are restricted to the ones of some special properties, we may be able to find a social decision rule which satisfies Arrow's five conditions. The results in the previous section and in this section show that the same is true for our formulation.

Footnotes

1. If the definition domain is the class of all subsets, the weak axiom of revealed preference is equivalent to the strong axiom. This is shown by Arrow in [2]. But if the definition domain is of the class of budget constraint sets, the weak axiom cannot imply the strong axiom. This is shown by Gale in [4].

2. For this relation, see [6].

3. We assume in the following that $m > 1$ and $n > 2$. If $m = 1$, we have no social choice problem. If $n = 2$, we have no problem of integrability condition. Thus, if the number of commodities is two, any welfare function has the desired property. In Arrow's case, the impossibility theorem holds for more than one commodity world. This is stated as a corollary (Possibility Theorem for Individualistic Assumptions), p. 63 in [1]. This difference is caused by the fact that our social choice function is defined on the sets in an n-dimensional space and Arrow's social choice function is defined on the sets in an $m \times n$-dimensional space.

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